On supersymmetry and other properties of a class of marginally deformed backgrounds

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Abstract

We summarize our recent work on supergravity backgrounds dual to part of the Coulomb branch of $\mathcal{N} = 1$ theories constructed as marginal deformations of $\mathcal{N} = 4$ Yang–Mills. In particular, we present a summary of the behaviour of the heavy quark-antiquark potential which shows confining behaviour in the IR as well as of the spectrum of the wave equation. The reduced supersymmetry is due to the implementation of T-duality in the construction of the deformed supergravity solutions. As a new result we analyze and explicitly solve the Killing spinor equations of the $\mathcal{N} = 1$ background in the superconformal limit.

1 Introduction

$\mathcal{N} = 4$ supersymmetric Yang–Mills admits a three parameter family of exactly marginal deformations with a global $U(1)^3$ symmetry preserving $\mathcal{N} = 1$ supersymmetry [1]. The gravity dual of the $\mathcal{N} = 4$ theory deformed by operators of the form Tr $[\Phi_1\{\Phi_2, \Phi_3\}]$ can be constructed using an $SL(2, \mathbb{R}) \in SL(3, \mathbb{R})$ symmetry of the complete $SL(3, \mathbb{R}) \times SL(2, \mathbb{R})$ duality group of type-IIB supergravity compactified on a two-torus [2] (see also [3]-[9]). When the deformation parameter is real, the marginally deformed background can be obtained through a sequence of T-duality transformations and coordinate shifts. As this derivation applies to supergravity backgrounds preserving at least a global symmetry group isomorphic to $U(1)^3$, we can extend it to find the marginal deformations of solutions away from the conformal point. The Coulomb branch of the $\mathcal{N} = 4$ theory arises when the $SO(6)$ scalar fields acquire non-vanishing expectation values. These Higgs expectation values correspond on the gravity side to multicenter distributions of branes. In [10] we employed a sequence of T-dualities and coordinates shifts on multicenter solutions to find gravity duals for the Coulomb branches of marginally deformed $\mathcal{N} = 4$ Yang–Mills. In this note we will focus on the deformation of uniform continuous distributions of D3-branes on a disc and on a three-dimensional spherical shell preserving a global $SO(4) \times SO(2)$ symmetry group. We will first briefly present the marginally deformed backgrounds. We will then investigate the supersymmetry preserved by these deformed solutions by explicitly constructing the Killing spinor. We also include a summary of the evaluation of the Wilson loop operator along the deformed background, and an analysis of the spectra of massless excitations after solving the corresponding Laplace equation. The reader should consult [10] for further details and related references.

2 Marginally deformed backgrounds

The supergravity solutions describing the Coulomb branch of $\mathcal{N} = 4$ supersymmetric Yang–Mills at strong 't Hooft coupling involve a metric and a self-dual 5-form as the only non-trivial fields. The metric has the form

$$ds_{10}^2 = H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}dx_i dx_i , \qquad \mu = 0, 1, 2, 3 , \quad i = 1, 2, \ldots, 6 , \qquad (1)$$

the self-dual 5-form is given by

$$F_5 = dA_4 + *_{10}dA_4 , \qquad (dA_4)_{0123i} = -\partial_i H^{-1} \qquad (2)$$

and the dilaton is a constant Φ_0. The solution is completely characterized by a harmonic function in $\mathbb{R}^6$. We are interested in the field theory limit in which the solution asymptotically becomes $AdS_5 \times S^5$ with each factor having radius $R = (4\pi g_s N)^{1/4}$, in string units.

In order to obtain the deformed background we first split the metric into a seven-dimensional piece, and a three-torus parametrized by some angles ϕ_i (with $\phi_i \in (0, 2\pi)$),

$$ds_{10}^2 = G_{IJ}(x)dx^I dx^J + \sum_{i=1}^{3} z_i(x)d\phi_i^2 , \qquad I = 1, 2, \ldots, 7 . \tag{3}$$

The seven-dimensional metric will not depend on the three angles. The self-dual 5-form is then given by

$$F_5 = dC^{(1)} \wedge d\phi_1 \wedge d\phi_2 \wedge d\phi_3 + \frac{1}{\sqrt{z_1 z_2 z_3}} *_7 dC^{(1)} , \tag{4}$$

for some 1-form $C^{(1)} = C_I^{(1)} dx^I$. Compatibility of (2) with (4) shows that there must be a 4-form $C^{(4)}$ such that $dC^{(4)} = *_7 dC^{(1)}/\sqrt{z_1 z_2 z_3}$. We now change variables through [2]

$$\phi_1 = \varphi_3 - \varphi_2 , \qquad \phi_2 = \varphi_1 + \varphi_2 + \varphi_3 , \qquad \phi_3 = \varphi_3 - \varphi_1 , \tag{5}$$

and perform a T-duality transformation along the φ_1 direction, a coordinate shift $\varphi_2 \to \varphi_2 + \gamma\varphi_1$, and again a T-duality along the φ_1 direction. The resulting background was found in [10]. For our purposes here we write the metric in the following form

$$ds_{10}^2 = G_{IJ}dx^I dx^J + G(z_2 + z_3)\left[d\varphi_1 + \frac{z_2}{z_2 + z_3}d\varphi_2 + \frac{z_2 - z_3}{z_2 + z_3}d\varphi_3\right]^2 \tag{6}$$

$$+ G\left(z_1 + \frac{z_2 z_3}{z_2 + z_3}\right)\left[d\varphi_2 - \frac{z_1(z_2 + z_3) - 2z_2 z_3}{z_1 z_2 + z_2 z_3 + z_3 z_1}d\varphi_3\right]^2 + 9\frac{z_1 z_2 z_3}{z_1 z_2 + z_2 z_3 + z_3 z_1}d\varphi_3^2 .$$

We will use a frame basis e^a, where the indexes running from $0, 1, \ldots, 6$ correspond to some frame basis for the seven-dimensional metric G_{IJ}, whereas $7, 8$ and 9 correspond to the natural frame read off from (6), i.e. $e^{i+6} = (\ldots)[d\varphi_i + \ldots]$. For the various forms and the dilaton supporting the solution we have

$$\begin{aligned} B &= \gamma(z_1 z_2 + z_2 z_3 + z_3 z_1)^{1/2} e^7 \wedge e^8 , \\ A^{(2)} &= 3\gamma d\varphi_3 \wedge C^{(1)} , \\ F_5 &= 3G dC^{(1)} \wedge d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 + \frac{1}{\sqrt{z_1 z_2 z_3}} *_7 dC^{(1)} , \\ e^{2\Phi} &= e^{2\Phi_0} G . \end{aligned} \tag{7}$$

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For notational convenience we have defined $G^{-1} = 1 + \gamma^2(z_1 z_2 + z_1 z_3 + z_2 z_3)$. We also note that the supergravity description remains valid at a generic point of the manifold if

$$R \gg 1 \qquad \text{and} \qquad \gamma R^2 \equiv \hat{\gamma} \ll R . \tag{8}$$

The latter condition is also sufficient for the 2-torus parametrized by $\varphi_{1,2}$ to remain much larger than the string scale after the T-dualities. Finally, note that the periodicities of the angular variables ϕ_i remain intact in the deformed background.

2.1 The $SO(4) \times SO(2)$ background

We will now present the marginally deformed $SO(4) \times SO(2)$ background in the case where the D3-branes are uniformly distributed on a disc of radius r_0. The original metric is of the form (1) with the flat metric in $\mathbb{R}^6$ given by

$$
\begin{aligned}
ds_{\mathbb{R}^6}^2 &= \frac{r^2 + r_0^2 \cos^2 \theta}{r^2 + r_0^2} \, dr^2 + (r^2 + r_0^2 \cos^2 \theta) d\theta^2 + (r^2 + r_0^2) \sin^2 \theta d\phi_1^2 \\
&\quad + r^2 \cos^2 \theta (d\psi^2 + \sin^2 \psi \, d\phi_2^2 + \cos^2 \psi \, d\phi_3^2) ,
\end{aligned}
\tag{9}
$$

where the 3-sphere line element is

$$d\Omega_3^2 = d\psi^2 + \sin^2 \psi \, d\phi_2^2 + \cos^2 \psi \, d\phi_3^2 \tag{10}$$

and the harmonic function reduces in this case to

$$H = \frac{R^4}{r^2(r^2 + r_0^2 \cos^2 \theta)} . \tag{11}$$

The forms necessary to compute the NS-NS and R-R field strengths are

$$
\begin{aligned}
C^{(1)} &= R^4 \frac{r^2 + r_0^2}{r^2 + r_0^2 \cos^2 \theta} \, \cos^4 \theta \sin \psi \cos \psi \, d\psi , \\
C^{(4)} &= -R^{-4} r^2 (r^2 + r_0^2 \cos^2 \theta) \, dt \wedge dx_1 \wedge dx_2 \wedge dx_3 .
\end{aligned}
\tag{12}
$$

The above allow to read off, with the aid of (3), the coordinates z_1, z_2 and z_3, and from them to compute

$$G^{-1} = 1 + \hat{\gamma}^2 \frac{\cos^2 \theta}{r^2 + r_0^2 \cos^2 \theta} \left[(r^2 + r_0^2) \sin^2 \theta + r^2 \cos^2 \theta \sin^2 \psi \cos^2 \psi \right] . \tag{13}$$

The case of a distribution of branes on the surface of a sphere of radius r_0 follows from analogous expressions, after we replace $r_0^2 \rightarrow -r_0^2$.

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3 Supersymmetry

We will now analyze in detail the supersymmetry of the marginally deformed backgrounds, and describe the origin of the reduced supersymmetry of the solutions. We will need the type-IIB supergravity Killing spinor equations corresponding to the gravitino and dilatino variations. In the string frame they read (see, for instance, [11])

$$D_\mu \epsilon - \frac{1}{8}\slashed{H}_\mu \sigma^3 \epsilon + \frac{1}{16}e^\Phi \sum_{n=1}^{5} \frac{1}{(2n-1)!}\, \slashed{G}^{(2n-1)}\Gamma_\mu P_n \epsilon = 0 \ ,$$

$$\slashed{\partial}\Phi\epsilon - \frac{1}{12}\slashed{H}\sigma^3 \epsilon + \frac{1}{4}e^\Phi \sum_{n=1}^{5} \frac{(n-3)}{(2n-1)!}\, \slashed{G}^{(2n-1)} P_n \epsilon = 0 \ , \tag{14}$$

where $P_n = \sigma_1$ for even n or $P_n = i\sigma_2$ when n is odd, with σ_i the Pauli matrices, and we have defined $G^{(2n+1)} = dA^{(2n)} - H \wedge A^{(2n-2)}$.

3.1 The supersymmetry breaking mechanism and T-duality

We first present the mechanism responsible for the supersymmetry breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 1$ when deforming the original background. We will concentrate on the background with $SO(4)\times SO(2)$ global symmetry in the disc case, and summarize the relevant results in [10]. The solution for the Killing spinor in the undeformed case can split into a part which is a singlet of the $U(1)$ rotations corresponding to the angles φ_1 and φ_2, and a part orthogonal to that. After the T-dualities and the coordinate shift only this part survives and remains a Killing spinor of the deformed theory. For any multicenter metric of the form (1) the Killing spinor is

$$\epsilon = H^{-1/8}\epsilon_0 \ , \tag{15}$$

with ϵ_0 a spinor subject to the projection

$$i\Gamma^{0123}\epsilon_0 = \epsilon_0 \ , \tag{16}$$

where the indexes refer to the directions along the brane. This projection will not be necessary in the conformal case in which $r_0 = 0$. The spinor ϵ_0 is determined to be

$$\epsilon_0 = e^{\frac{1}{2}f(r,\theta)\Gamma_{12}}e^{\frac{\psi}{2}\Gamma_{13}}e^{\frac{1}{2}\phi_i\sigma_i}\bar{\epsilon}_0 \ , \tag{17}$$

where $\bar{\epsilon}_0$ is a constant spinor and where we have defined

$$f(r,\theta) = \tan^{-1}\left(\frac{r\tan\theta}{(r^2 + r_0^2)^{1/2}}\right) \quad \text{and} \quad \sigma_1 = \Gamma_{24} \ , \quad \sigma_2 = \Gamma_{35} \ , \quad \sigma_3 = \Gamma_{16} \ . \tag{18}$$

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The calculations leading to the above expressions have been performed in a frame which is read off directly from (9) (including the harmonic function), that is $e^1 = (\ldots)dr$, $e^2 = (\ldots)d\theta$, $e^3 = (\ldots)d\psi$ and $e^{i+3} = (\ldots)d\phi^i$. After rewriting the spinor ϵ_0 in the φ_i coordinate system and a simple computation we find that the required spinor invariant under variations of φ_1 and φ_2 is given by

$$\epsilon_{0,\text{inv}} = e^{\frac{1}{2}f(r,\theta)\Gamma_{12}} e^{\frac{\psi}{2}\Gamma_{13}} e^{\frac{3}{2}\sigma_3\varphi_3} \bar{\epsilon}_{0,\text{inv}} \; . \tag{19}$$

The constant spinor $\bar{\epsilon}_{0,\text{inv}}$ in terms of $\bar{\epsilon}_0$ is given by

$$\bar{\epsilon}_{0,\text{inv}} = \frac{1}{4}(\mathbb{1} - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3)\bar{\epsilon}_0 \; , \tag{20}$$

where the prefactor acts as a projector and by construction we have

$$\sigma_1\bar{\epsilon}_{0,\text{inv}} = \sigma_2\bar{\epsilon}_{0,\text{inv}} = \sigma_3\bar{\epsilon}_{0,\text{inv}} \; . \tag{21}$$

3.2 The explicit Killing spinor in the conformal case

We will now construct the ten-dimensional Killing spinor in the deformed background. This computation is quite involved technically, so that for simplicity we will only consider in some detail the conformal limit, $r_0 = 0$. The non-vanishing components of the spin connection for (6) are

$$
\begin{aligned}
\omega^{i4} &= e^i \; , \quad i = 1,2,3 \; , \quad \omega^{56} = \frac{\text{s}_\theta}{\text{c}_\theta} e^6 \; , \\[2mm]
\omega^{57} &= G\frac{\text{s}_\theta}{\text{c}_\theta}\left[1 + \hat{\gamma}^2 \text{c}_\theta^4\left(1 - \text{s}_\psi^2\text{c}_\psi^2\right)\right] e^7 \; , \\[2mm]
\omega^{58} &= -\frac{G\text{s}_\theta\text{c}_\theta^3}{u^2}\left[1 - \text{s}_\psi^2\text{c}_\psi^2 + \hat{\gamma}^2\frac{u^2}{\text{c}_\theta^2}\right] e^8 + \frac{\sqrt{G}\text{s}_\psi\text{c}_\psi\text{c}_\theta^2}{u^2} e^9 \; , \\[2mm]
\omega^{59} &= \frac{\sqrt{G}\text{s}_\psi\text{c}_\psi\text{c}_\theta^2}{u^2} e^8 + \frac{\text{c}_\theta}{\text{s}_\theta}\frac{\text{s}_\theta^4 - \text{c}_\theta^4\text{s}_\psi^2\text{c}_\psi^2}{u^2} e^9 \; , \\[2mm]
\omega^{67} &= \frac{1}{4}\hat{\gamma}^2 G\text{c}_\theta^3\text{s}_{4\psi}e^7 - \frac{\text{s}_\psi\text{c}_\psi\text{c}_\theta}{u} e^8 + \frac{\sqrt{G}\text{s}_\theta}{u} e^9 \; , \\[2mm]
\omega^{68} &= -\frac{\text{s}_\psi\text{c}_\psi\text{c}_\theta}{u} e^7 - \frac{G\text{c}_\theta^3\text{s}_{4\psi}}{4u^2} e^8 - \frac{\sqrt{G}\text{c}_{2\psi}\text{s}_\theta\text{c}_\theta^2}{u^2} e^9 \; , \\[2mm]
\omega^{69} &= -\frac{\sqrt{G}\text{s}_\theta}{u} e^7 - \frac{\sqrt{G}\text{c}_{2\psi}\text{s}_\theta\text{c}_\theta^2}{u^2} e^8 - \frac{2\text{s}_\theta^2\text{c}_\theta\text{c}_{2\psi}}{\text{s}_{2\psi}}\frac{1}{u^2} e^9 \; , \\[2mm]
\omega^{78} &= -\frac{\text{s}_\psi\text{c}_\psi\text{c}_\theta}{u} e^6 \; , \quad \omega^{79} = -\frac{\sqrt{G}\text{s}_\theta}{u} e^6 \; , \\[2mm]
\omega^{89} &= \frac{\sqrt{G}\text{s}_\psi\text{c}_\theta^2}{u^2} e^5 - \frac{\sqrt{G}\text{c}_{2\psi}\text{s}_\theta\text{c}_\theta^2}{u^2} e^6 \; ,
\end{aligned}
\tag{22}
$$

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where we have introduced the notation $c_\alpha \equiv \cos\alpha$ and $s_\alpha \equiv \sin\alpha$ and

$$u \equiv c_\theta \sqrt{s_\theta^2 + c_\theta^2 s_\psi^2 c_\psi^2} \ . \tag{23}$$

From the gravitino variation along the brane directions and $\mu = r$ we get the following expression for the spinor [12]

$$\epsilon = e^{-\frac{1}{2}\ln r \Gamma^{(5)}A} \left[\mathbb{1} - \frac{1}{2}x^\alpha \Gamma_{\alpha 5}\left(\mathbb{1} + \Gamma^{(5)}A\right) \right] \eta \ , \tag{24}$$

where η is a spinor that could depend on everything but the worldvolume coordinates x^α and r. Also $\Gamma^{(5)} = i\Gamma^{0123}$ and A is defined as

$$A \equiv \sqrt{G}\left(\mathbb{1} - \hat{\gamma}u\Gamma^{78}*\right) = e^{-\tan^{-1}(\hat{\gamma}u)\Gamma^{78}*} \ , \tag{25}$$

with $*$ the complex conjugation operator. The gravitino variations along $\mu = \theta$, $\mu = \psi$ and $\mu = 9$ determine

$$\eta = e^{\frac{1}{2}\tan^{-1}\hat{\gamma}u\Gamma^{78}*}e^{-\frac{1}{2}\tan^{-1}(2\tan\theta/\sin 2\psi)\Gamma_{89}}e^{-\frac{1}{2}\theta\Gamma_{45}\Gamma^{(5)}}e^{\frac{1}{2}\psi(\Gamma_{78}-\Gamma_{46}\Gamma^{(5)})}e^{\frac{3}{2}\Gamma_{68}\varphi_3}\eta_0 \ , \tag{26}$$

where η_0 is a constant spinor satisfying two projections arising from the supersymmetry variation for the dilatino,

$$\Gamma_{47}\Gamma^{(5)}\eta_0 = \Gamma_{68}\eta_0 = \Gamma_{59}\eta_0 \ . \tag{27}$$

In addition, from the gravitino variations along $\mu = 7$ and $\mu = 8$ we conclude that there is no dependence of the spinor on φ_1 and φ_2. Note that the spinor (26) does not reduce precisely to that in (19) as $\hat{\gamma} \to 0$, since the corresponding frames are different.

4 Wilson loops

The marginally deformed backgrounds in the Coulomb branch contain a very rich structure. In [10] we probed the geometry of the deformation by evaluating the expectation value of the Wilson loop operator along the transversal space to the worldvolume of the branes. The Wilson loop can be computed by minimizing the Nambu–Goto action for a fundamental string in a given supergravity background [13]. We will in particular take the string to stretch along a trajectory given by

$$\theta = 0 \ , \qquad \psi = \frac{\pi}{4} \ , \qquad \phi_2 = \phi_3 \equiv \phi \ , \qquad x_{2,3} = \text{constant} \ , \tag{28}$$

which is consistent with the equations of motion if the conserved angular momenta coincide, $l_{\phi_2} = l_{\phi_3} \equiv l$. In the absence of a Higgs expectation value, $r_0 \to 0$, we recover

the expected Coulombic behaviour for the heavy quark-antiquark potential. In the non-conformal case we will only concentrate on the case of the disc distribution. Then the behaviour of the quark-antiquark potential depends on the relation between the various parameters of the theory. At a finite value of the length of the loop, $L_{\text{fin}} = \frac{\pi R^2}{r_0}\sqrt{1-l^2}$, the potential goes to zero,

$$E_{q\bar{q}} \simeq -\frac{1-(1+\hat{\gamma}^2/4)l^2}{4(1-l^2)} \, r_0^3 \left(\frac{L_{\text{fin}}-L}{\pi R^2}\right)^2 \, . \tag{29}$$

If $(1+\hat{\gamma}^2/4)l^2 < 1$ the potential vanishes monotonically. When $(1+\hat{\gamma}^2/4)l^2 > 1$ the potential becomes positive, reaches a maximum and then becomes zero again. However, beyond the maximum the force between the quark and the antiquark becomes repulsive. To avoid this unphysical region we take the limit of a large deformation parameter, $\hat{\gamma} \gg 1$, and simultaneously take the angular parameter $l \to 1$. This introduces a hierarchy of widely separated scales and in particular we note that

$$\frac{r_0}{\hat{\gamma}} \ll r_0 \ll \frac{r_0}{\sqrt{1-l^2}} \, . \tag{30}$$

The above limit avoids the unphysical region by removing the piece in the deep IR. The conformal region is also removed by avoiding probing with energies in the deep UV, that is extremely higher than the Higgs expectation value. In this way, for small values of the separation distance $\bar{L} = \frac{L}{R^2\sqrt{1-l^2}}$, we find a linear behaviour

$$E_{q\bar{q}} \simeq \frac{r_0^2}{2\pi}\bar{L} \, , \quad \text{as} \quad \frac{\sqrt{1-l^2}}{r_0} \ll \bar{L} \ll \frac{1}{r_0} \tag{31}$$

and a logarithmic dependence for large values of $\bar{L}$,

$$E_{q\bar{q}} \simeq \frac{r_0}{\pi}\ln\left(r_0\bar{L}\right) \, , \quad \text{as} \quad \frac{1}{r_0} \ll \bar{L} \ll \frac{\hat{\gamma}}{r_0} \, . \tag{32}$$

A logarithmic form for a confining potential instead of a linear one was suggested long time ago in order to explain quarkonium spectra with energy level spacing independent of the particle mass [14]. The various different behaviours are depicted in Figure 1.

5   The wave equation

The study of the massless field equations in a supergravity background is a problem of great interest because the spectrum of fluctuations after the AdS/CFT correspondence corresponds to gauge theory operators. In [10] we found that the marginal deformation
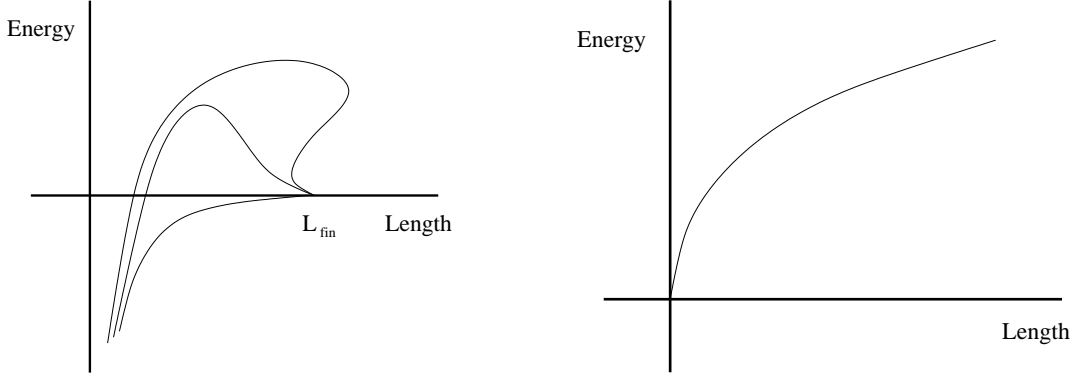
Figure 1: The energy as a function of the quark-antiquark separation distance L. <u>Left</u>: As the value of $\hat{\gamma}$ increases, the shape of $E_{q\bar{q}}$ is modified from the lower to the upper curve. The result is trusted until the maximum energy is reached. <u>Right</u>: In the limit $\hat{\gamma} \to \infty$, $l \to 1$ and for large separations we find the logarithmic confining dependence (32). For smaller separations the behaviour is linear, (31).

in the massless scalar field equation arises as an additional term for the corresponding undeformed equation,

$$\Box \Psi = \Box_{\gamma=0} \Psi + \gamma^2 \lambda_{ij} \ \partial_i \partial_j \Psi = 0 \quad , \tag{33}$$

where we have introduced the symmetric matrix

$$\lambda_{ij} \equiv \frac{z_1 z_2 + z_1 z_3 + z_2 z_3}{z_i} \delta_{ij} - \frac{z_1 z_2 z_3}{z_i z_j} \ . \tag{34}$$

5.1 The undeformed case

It turns out that in the absence of deformation, the Laplace equation provides a set of differential equations that can be solved by transforming them into Schrödinger equations whose potential is determined by the geometry of the supergravity background. We first separate variables on the Laplace operator through the plane-wave ansatz

$$\Psi = \frac{1}{(2\pi)^{5/2}} e^{ik \cdot x} e^{in\phi_1} \Psi_{S^3}(\psi, \phi_2, \phi_3) \psi^{(1)}(\theta) \psi^{(2)}(r) \ , \qquad n \in \mathbb{Z} \ . \tag{35}$$

The equation for Ψ_{S^3} is the usual eigenvalue equation on S^3 with eigenvalue $-l(l+2)$, $l = 0, 1, \ldots$. The one for $\psi^{(1)}(\theta)$ becomes the Jacobi differential equation provided that a separation variable E is quantized as

$$E_{m,l,n} = (l + |n| + 2m)(l + |n| + 2m + 4) \ , \qquad m = 0, 1, \ldots \ . \tag{36}$$

This parameter enters into the radial equation for $\psi^{(2)}(r)$ which is the only one sensitive to the details of the geometry. We must therefore consider three possible cases.

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The conformal limit: Once we set $r_0 = 0$ the differential equation for $\psi^{(2)}(r)$ can be transformed into a Schrödinger equation with potential

$$V(z) = \frac{15/4 + E_{m,l,n}}{z^2} \tag{37}$$

and eigenvalue $M^2 R^4$, where $z = 1/r$. As this is a positive definite potential, which vanishes for large values of z, we find a continuous spectrum with no mass gap.

The disc: When the D3-branes distribute uniformly on a disc of radius r_0 we transform the problem into a Schrödinger equation with potential

$$V(z) = (l+1)^2 - \frac{n^2 - 1/4}{\cosh^2 z} + \frac{15/4 + E_{m,l,n}}{\sinh^2 z} \ , \tag{38}$$

with $\sinh z = r_0/r$ and eigenvalue $M^2 R^4/r_0^2$. This potential decreases monotonically from arbitrarily large positive values to the constant $(l+1)^2$, as z varies from 0 to ∞, and belongs to the family of Pöschl–Teller potentials in quantum mechanics of type II. We therefore find a continuous spectrum with mass gap given by

$$M_{\text{gap},l} = (l+1)\frac{r_0}{R^2} \ . \tag{39}$$

It turns out that this gap is $(l+1)^2$ degenerate.

The sphere: In this case, after changing to a new radial variable $\sin z = r_0/r$, we transform the problem into a Schrödinger equation with potential

$$V(z) = -(l+1)^2 + \frac{n^2 - 1/4}{\cos^2 z} + \frac{15/4 + E_{m,l,n}}{\sin^2 z} \ , \tag{40}$$

and eigenvalue $M^2 R^4/r_0^2$, which belongs to the family of Pöschl–Teller potentials of type I. The mass eigenvalue turns out to be quantized as

$$M_{k,m,l,n}^2 = 4(k + m + |n| + 1)(k + m + |n| + l + 2)\frac{r_0^2}{R^4} \ , \tag{41}$$

with degeneracy $(l+1)^2(k + m + |n| + 1)^2$.

5.2   The deformed case

The deformed backgrounds include the second term on the right hand side of (33). As this term breaks the $SO(4)$ spherical symmetry unless we focus on solutions independent of the angles ϕ_2 and ϕ_3, we must now consider the ansatz (35) but with Ψ_{S^3} having no dependence on ϕ_2, ϕ_3 (a slight generalization can be found in [10]). It turns out that then the quantum number l has to be an even integer, so that in the rest we replace it

by $2l$. The radial equation for $\psi^{(2)}(r)$ is basically the same as in the undeformed case. It is the equation for $\psi^{(1)}(\theta)$ that receives the major effect of the deformation. As before the equation can be transformed into a Schrödinger equation with potential

$$V(\theta) = -4 + \frac{n^2 - 1/4}{\sin^2 \theta} + \frac{4l(l+1) + 3/4}{\cos^2 \theta} + n^2 \hat{\gamma}^2 \cos^2 \theta . \tag{42}$$

When $n = 0$ we recover the undeformed limit. Solving the equation for $n \neq 0$ is a difficult task, because there is a change in the nature of the singularity at infinity of the corresponding differential equation which is of the Fuchsian type. Perturbation theory or an asymptotic expansion can still be used, for small or large values of $n^2 \hat{\gamma}^2$, respectively. For arbitrary values of the deformation parameter we may appropriately redefine $\psi^{(1)}$ (for details we refer the reader to [10]) to reduce the problem to a confluent form of the Heun differential equation (see, for instance, [15]). The Heun differential equation is known to be related to the BC_1 Inozemtsev system, which is a one-particle quantum mechanical model in one dimension with an elliptic potential [16]. In fact, after a trigonometric limit the Inozemtsev system becomes a Schrödinger problem with a trigonometric Pöschl–Teller potential, which is nothing but (42) with $\hat{\gamma} = 0$. The complete potential can still be recovered through a generalized form of the trigonometric limit [17, 10]. The importance of the relation to the Inozemtsev system is that this is an integrable model. Therefore, the Bethe ansatz method can be used to find solutions to the deformed differential equation. We expect that further progress can be made along this line.

6   Conclusions

The exactly marginal deformations of $\mathcal{N} = 4$ supersymmetric Yang–Mills by operators of the form $\text{Tr}\,[\Phi_1\{\Phi_2, \Phi_3\}]$ amount on the gravity side of the AdS/CFT correspondence to a deformation on the transversal space to the worldvolume of the branes. We have explored the geometry of this deformation both in the conformal limit and for a continuous distribution of D3-branes with $SO(4) \times SO(2)$ global symmetry group. We have in particular presented the mechanism responsible for the breaking of supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 1$ as the supergravity background gets deformed, and constructed the ten-dimensional Killing spinor. By evaluating the Wilson loop operator we have found regimes where the quark-antiquark interaction is completely screened, or where a confining behaviour arises. Finally, we have also performed a detailed analysis of the spectra of massless excitations in the deformed $SO(4) \times SO(2)$ background. We have

found solutions to the Laplace equation by transforming the corresponding differential equations into Schrödinger problems. The rich structure and underlying geometry of the deformations present themselves as a promising path for further investigations.

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